Statistical Learning II

Lecture 6 - Bias-Variance decomposition

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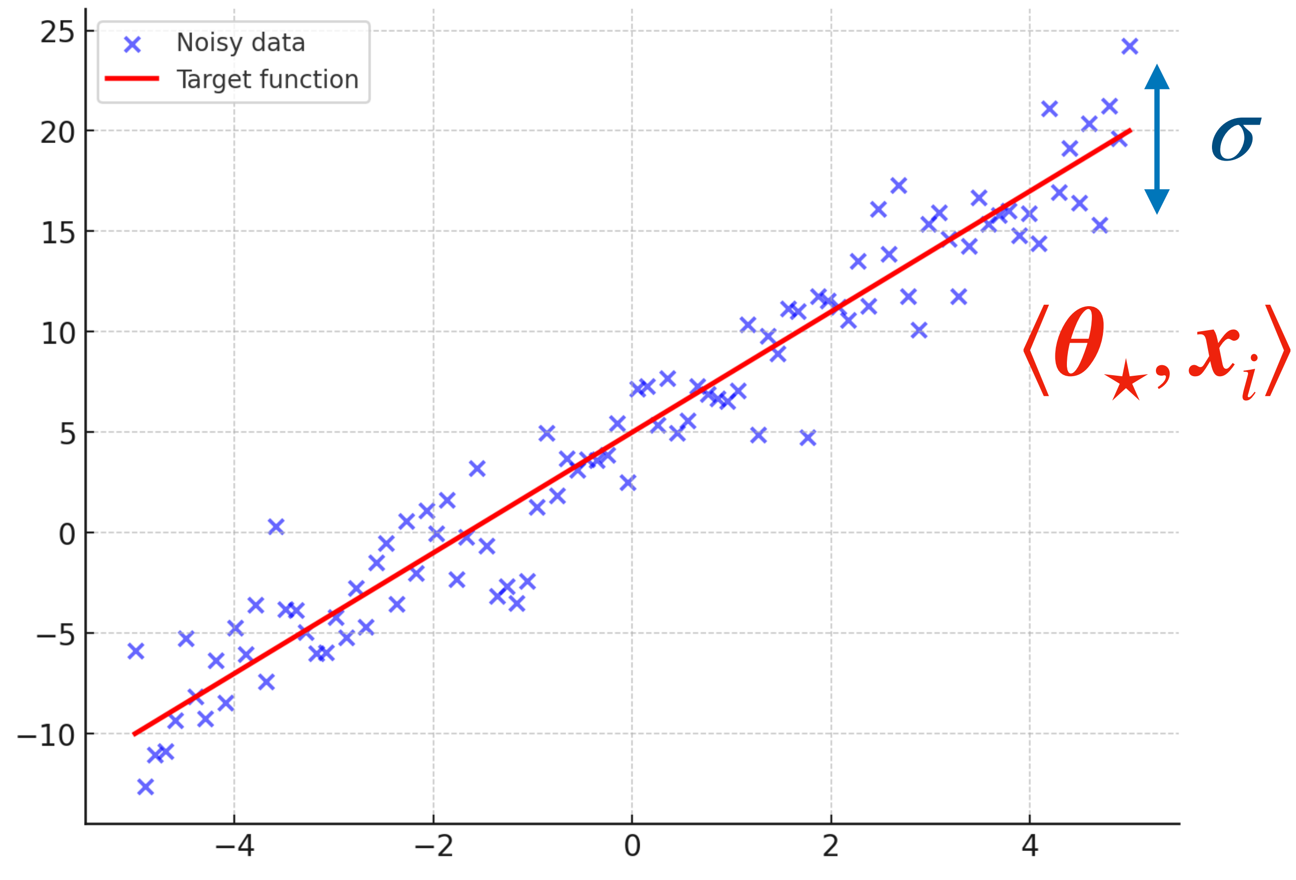
Assumptions

We now assume the following data generative model:

$$y_i = \langle \boldsymbol{\theta}_\star, \boldsymbol{x}_i \rangle + \varepsilon_i$$

With:
- Fixed $\boldsymbol{\theta}_\star \in \mathbb{R}^d$ and $\boldsymbol{x}_i \in \mathbb{R}^d$ "fixed design"
- $\mathbb{E}[\varepsilon_i] = 0$ and $\mathbb{E}[\varepsilon_i^2] = \sigma^2 < \infty$



Decomposition of OLS

Assume $X \in \mathbb{R}^{n\times d}$ is full-rank and $n > d$:

$$\hat{\boldsymbol{\theta}}_{OLS}(\boldsymbol{X}, \boldsymbol{y}) = \boldsymbol{\theta}_\star + \frac{1}{n}\hat{\boldsymbol{\Sigma}}_n^{-1}\boldsymbol{X}^\top \boldsymbol{\varepsilon}$$

"signal" "noise"

In particular:

- Bias: $\mathbb{E}_{\boldsymbol{\varepsilon}}\left[\hat{\boldsymbol{\theta}}_{OLS}(\boldsymbol{X}, \boldsymbol{y})\right] = \boldsymbol{\theta}_\star$ "Unbiased"
- Variance: $\mathrm{Var}_{\boldsymbol{\varepsilon}}\left[\hat{\boldsymbol{\theta}}_{OLS}(\boldsymbol{X}, \boldsymbol{y})\right] = \frac{\sigma^2}{n}\hat{\boldsymbol{\Sigma}}_n^{-1}$

Informally, if $\hat{\boldsymbol{\Sigma}}_n \to \boldsymbol{\Sigma}$ a rank d matrix as $n \to \infty$, then:

$\hat{\boldsymbol{\theta}}_{OLS} \to \boldsymbol{\theta}_\star$ as $n \to \infty$ "Consistency"

Risk of OLS

Therefore, we have the following final result for the excess risk of OLS

$$\mathbb{E}_{\boldsymbol{\varepsilon}}\left[\mathscr{R}(\hat{\boldsymbol{\theta}}_{OLS})\right] - \sigma^2 = \sigma^2 \frac{d}{n}$$

Remarks:

- Excess risk is proportional to the noise level $\mathbb{E}[\varepsilon^2] = \sigma^2$.
- Excess risk is proportional to the data dimension.
- To achieve excess risk $\Delta\mathscr{R} < \delta$, need:

$$n > \frac{\sigma^2 d}{\delta}$$

samples.